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SOLITARY WAVES IN A WEAKLY STRATIFIED TWO-LAYER FLUID

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A problem on stationary waves on the interface between a homogeneous fluid and an exponentially stratified fluid is considered. The density difference on the interface being assumed to have the same order of smallness as the density gradient of the fluid inside the stratified layer, an equation of the second-order shallow water approximation is derived for description of propagation of finite-amplitude solitary waves.

Key words: *two-layer fluid, weak stratification, solitary waves.*

Introduction. A method for deriving equations of the second-order approximation of the shallow water theory with the use of partial expansion of the sought quantities with respect to the powers of the small parameter was proposed by Ovsyannikov [1, 2]. In [2], this method was used to derive a differential equation that describes finite-amplitude stationary long waves in a two-layer fluid with constant densities in the layers. Extensions of this approximation to solitary waves in a fluid with piecewise-exponential stratification were considered in [3, 4]. Owing to the presence of continuous weak stratification inside the flow, there appears an additional small parameter σ, which characterizes the density gradient in the fluid outside the picnocline modeled by the density jump surface. It was found that the order of σ with respect to another small parameter (dimensionless density jump μ on the interface between the layers) exerts a significant effect on the form of the equation for the principal term of long-wave asymptotics. Voronovich [3] considered the case where the quantities σ and μ had an identical order of smallness, which corresponded to a weakly expressed picnocline. The exact nonlinear equation for the stream function in the initial hydrodynamic formulation was approximated by a linearized equation. An asymptotic analysis of the Euler equations for an inhomogeneous fluid in the case of a picnocline with a small value of the ratio σ/μ was performed in [4] in the full nonlinear formulation. It was demonstrated, in particular, that the Ovsyannikov equation was obtained as a result of the limit transition $\sigma \to 0$ with a fixed nonzero value of the parameter μ.

In the present paper, we consider a particular case $\sigma/\mu \sim 1$ with no simplifying assumptions about the form of the initial equation for the stream function in the stratified layer. The resultant model equation for nonlinear long waves with dispersion in a weakly stratified two-layer fluid does not reduce to available approximations and possesses some interesting features.

1. Formulation of the Problem. We consider a two-dimensional stationary flow of a two-layer fluid in a domain bounded by a smooth horizontal bottom $y = -h_1$ and an impermeable top cover $y = h_2$ (Fig. 1). In the absence of wave motion, the unknown interface between the layers $y = \eta(x)$ is assumed to be in equilibrium at $y = 0$. For flows with solitary waves, the velocity vector of the fluid (u, v) in the jth layer $(j = 1, 2)$ is supposed to tend to a constant vector $(u_j, 0)$ as $x \to \pm\infty$ (u_j is the phase velocity of the wave with respect to the corresponding layer). The expression for the fluid density in the unperturbed flow is defined as

$$\rho_\infty(y) = \begin{cases} \rho_1, & -h_1 < y < 0, \\ \rho_2 \exp(-N^2 y/g), & 0 < y < h_2, \end{cases}$$

where $N = \text{const}$ is the Brunt–Väisälä frequency, and the constants $\rho_2 < \rho_1$ are the limit values of density above

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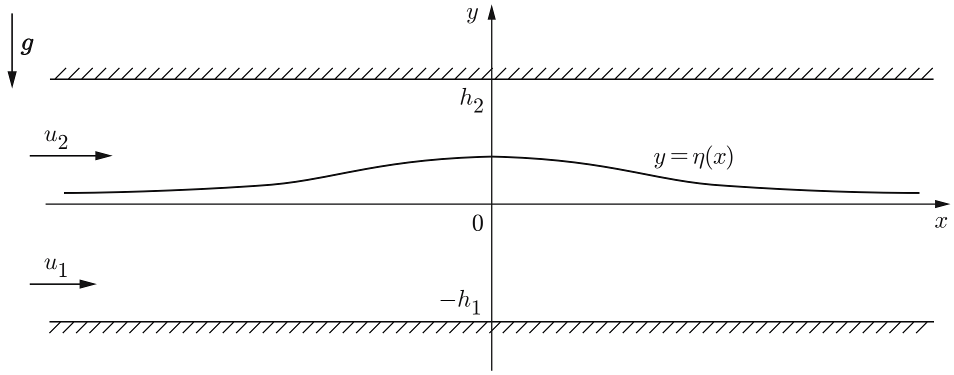

Fig. 1. Fluid motion.

and below the interface. The above-noted stratification is determined by the dimensionless Boussinesq parameters

$$\sigma = N^2 h_2/g, \qquad \mu = (\rho_1 - \rho_2)/\rho_2,$$

and the phase velocities of the wave are characterized by the density-based Froude numbers F_j:

$$F_j^2 = \frac{\rho_j\, u_j^2}{g(\rho_1 - \rho_2)h_j} \qquad (j = 1, 2).$$

Moreover, the problem involves a dimensionless geometric parameter $r = h_1/h_2$, which is the ratio of the unperturbed depths of the layers.

The fluid motion is defined if the stream function ψ_j for the velocity field $u = \psi_{jy}$, $v = -\psi_{jx}$ is known in the jth layer. Choosing the unperturbed depth of the upper layer h_2 as the linear scale and the flow rate of the fluid in the corresponding layer as the scale for the stream function $\psi = \psi_j$, we introduce the dimensionless variables

$$(x, y, \eta) = h_2(\bar{x}, \bar{y}, \bar{\eta}), \qquad \psi_j = u_j h_j \bar{\psi}_j \qquad (j = 1, 2).$$

Then, the stream function in the lower layer has to be a harmonic function:

$$\begin{gathered} \psi_{1xx} + \psi_{1yy} = 0 \qquad (-r < y < \eta(x)), \\ \psi_1(x, -r) = -1, \qquad \psi_1(x, \eta(x)) = 0 \end{gathered} \tag{1}$$

(hereinafter, the bar in the dimensionless quantities $\bar{x}$, $\bar{y}$, $\bar{\eta}$, and $\bar{\psi}_j$ is omitted). The stream function in the upper layer, because of exponential stratification of the latter, has to be a solution of a boundary-value problem for the nonlinear Dubreil-Jacotin–Long equation (see [4])

$$\begin{gathered} \psi_{2xx} + \psi_{2yy} + \lambda^2(\psi_2 - y) = \sigma(\psi_{2x}^2 + \psi_{2y}^2 - 1)/2 \qquad (\eta(x) < y < 1), \\ \psi_2(x, \eta(x)) = 0, \qquad \psi_2(x, 1) = 1, \end{gathered} \tag{2}$$

where $\lambda = Nh_2/u_2$. The dimensionless parameter λ is not independent and is related to σ, μ, and F_2 by the expression

$$\lambda^2 = \sigma/(\mu F_2^2). \tag{3}$$

The requirement of pressure continuity everywhere in the domain of the two-layer flow yields a condition that relates the derivatives ψ_1 and ψ_2 on the interface:

$$rF_1^2(r^2\psi_{1x}^2 + r^2\psi_{1y}^2 - 1) + 2\eta = F_2^2(\psi_{2x}^2 + \psi_{2y}^2 - 1) \qquad [y = \eta(x)]. \tag{4}$$

By virtue of Eqs. (1), (2), the boundary condition (4) is equivalent to the integral relation

$$\mu r^3 F_1^2 \int\limits_{-r}^{\eta} (\psi_{1y}^2 - \psi_{1x}^2)\, dy - (1+\mu)\eta^2 + \mu r F_1^2 (\eta - r)$$

$$+ \int\limits_{\eta}^{1} \mathrm{e}^{-\sigma\psi_2} [\mu F_2^2 (1 + \psi_{2y}^2 - \psi_{2x}^2) - 2\sigma^{-1}(\mathrm{e}^{\sigma\psi_2} - 1) + 2(\psi_2 - y)]\, dy$$

$$= 2\mu F_2^2 + 2(\lambda^{-2} + \sigma^{-2})(1 - \sigma - \mathrm{e}^{-\sigma}). \qquad (5)$$

This relation is the integral conservation law for the horizontal momentum flux of the fluid, written in terms of the stream function. It seems reasonable to use the boundary condition (4) to analyze the dispersion properties of the equations considered and to use the integral of equations of motion (5) to construct the long-wave asymptotic of the solution.

2. Long-Wave Approximation. Assuming the Boussinesq parameters σ and μ to be small quantities of the same order, we assume that $\mu = \sigma$ to simplify our calculations and use σ as a modeling parameter. In this case, according to Eq. (3), the parameter λ in Eqs. (2) and (5) has to be set equal to $\lambda = 1/F_2$. In deriving the long-wave approximation model, we use an asymptotic presentation of the stream functions in the form

$$\psi_j(x, y) = \psi_j^{(0)}(\xi, y) + \sigma \psi_j^{(1)}(\xi, y) + O(\sigma^2) \qquad (j = 1, 2), \qquad (6)$$

where $\xi = \sqrt{\sigma}\, x$ is a slow variable. The coefficients of expansion (6) are found by integrating the recurrent sequence [obtained from Eqs. (1) and (2)] of ordinary differential equations with an independent variable y, which involve ξ as a parameter. In particular, the lowest-order coefficients are expressed via the function η by the formulas

$$\psi_1^{(0)} = \frac{y - \eta}{r + \eta}, \qquad \psi_2^{(0)} = y - \eta\, \frac{\sin\alpha(y)}{\sin\alpha(\eta)},$$

where

$$\alpha(y) = (1 - y)/F_2, \qquad \alpha(\eta) = (1 - \eta)/F_2. \qquad (7)$$

Substituting series (6) into the integral relation (5) and leaving terms with accuracy to $O(\sigma^2)$, we obtain an equation of the second-order shallow water approximation for the function η, which describes the sought shape of the interface. Coming back to the dimensionless independent variable x in the principal order with respect to σ, we obtain the ordinary differential equation

$$\Big(\frac{d\eta}{dx}\Big)^2 = \eta^2 \cos^2\Big(\frac{\alpha(\eta)}{2}\Big)\, \frac{P(\eta; F_1, F_2)}{Q(\eta; F_1, F_2)}, \qquad (8)$$

where

$$P(\eta; F_1, F_2) = 3r(F_1^2 - 1) + 3F_2(r + \eta)\cot\,\alpha(\eta) - 3\eta - \eta^2,$$

$$Q(\eta; F_1, F_2) = q_0 + q_1\eta + q_2\eta^2 + q_3\eta^3 + q_4\eta^4,$$

q_0, q_1, q_2, q_3, and q_4 are the coefficients:

$$q_0 = (r^2F_1^2/3)\sin^2\alpha + (rF_2^2/4)(2 - F_2\sin 2\alpha),$$

$$q_1 = F_2 r\cot\,\alpha - (F_2^2/4)(2r\cos 2\alpha - 2 + 4r + F_2\sin 2\alpha),$$

$$q_2 = -F_2^2(1 + (1/2)\cos 2\alpha) + F_2[((2 - 3r)/2)\cot\,\alpha + (1/4)r\sin 2\alpha] + (r/2)\cot^2\alpha,$$

$$q_3 = (1/4)\cot\,\alpha\,[F_2(4 - \cos 2\alpha) - 2(r - 1)\cot\,\alpha], \qquad q_4 = -(1/2)\cot^2\alpha,$$

and $\alpha = \alpha(\eta)$ is determined by the second formula in Eqs. (7).

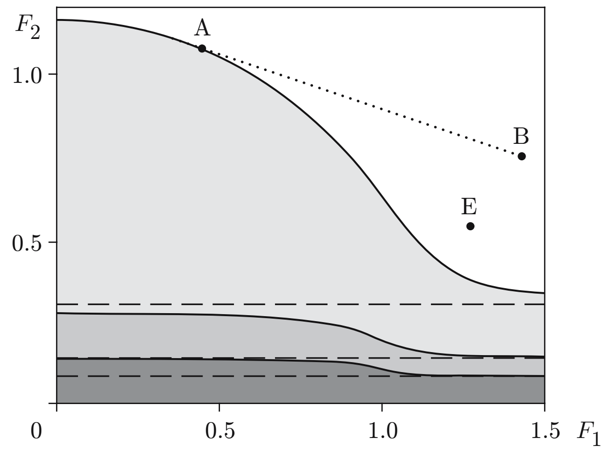

Fig. 2. Spectrum of the principal mode ($\sigma = \mu = 0.09$).

The solutions of Eq. (8) of the solitary-wave type are described by the quadratures

$$x = \pm \int_\eta^a \sqrt{\frac{Q(s; F_1, F_2)}{P(s; F_1, F_2)}} \frac{ds}{s \cos(\alpha(s)/2)}, \qquad (9)$$

where the wave amplitude is determined by the root $s = a$ of the function $P(s; F_1, F_2)$, which is closest to the point $s = 0$ (under the condition that this root is simple). In the neighborhood of the value $s = 0$, the function Q is positive, which follows from the relations

$$Q(0; F_1, F_2) = q_0\Big|_{\eta=0} = (r^2 F_1^2/3) \sin^2(1/F_2) + (rF_2^2/4)(2 - F_2 \sin(2/F_2)) > 0.$$

Therefore, for real solutions of the form (9) to exist, the inequality

$$P(0; F_1, F_2) = 3r(F_1^2 + F_2 \cot(1/F_2) - 1) > 0$$

should be satisfied, which can be interpreted in terms of dispersion properties of the initial problem. Equations (1), (2), and (4) linearized with respect to small perturbations of the one-dimensional piecewise-constant flow [wave packets with the function $\eta(x) = a \exp(ikx)$] yield a dispersion relation for the Froude numbers F_j and the wave number k:

$$\Delta(k; F_1, F_2) = 0. \qquad (10)$$

For $F_2^{-2} \geq k^2 + \sigma^2/4$, the dispersion function Δ has the form

$$\Delta = F_1^2 rk \coth(rk) + F_2^2\Big(\sqrt{1/F_2^2 - k^2 - \sigma^2/4} \cot \sqrt{1/F_2^2 - k^2 - \sigma^2/4} - \sigma/2\Big) - 1;$$

for $F_2^{-2} < k^2 + \sigma^2/4$, the dispersion function is defined by the analytical continuation of the expression given above. The spectrum of phase velocities of linear harmonic waves consists of points in the plane (F_1, F_2), for which the dispersion relation (10) has real roots k. Depending on the number of real wavenumbers corresponding to a specified pair of the Froude numbers (F_1, F_2), the spectrum is divided into modal regions whose boundaries are defined by the branches of the curve

$$\Delta(0; F_1, F_2) \equiv F_1^2 + F_2^2\Big(\sqrt{1/F_2^2 - \sigma^2/4} \cot \sqrt{1/F_2^2 - \sigma^2/4} - \sigma/2\Big) - 1 = 0.$$

The hatched regions in Fig. 2 correspond to the spectra of the first three modes. For points (F_1, F_2) outside the spectrum, the dispersion relation (10) does not generate wave packets oscillating with respect to x. Therefore, by analogy with surface waves in a homogeneous fluid, the supplement to the spectrum is interpreted as a region of supercritical flows. Taking into account this remark and the relation

$$P(0; F_1, F_2) = 3r\, \Delta(0; F_1, F_2) + O(\sigma^2)$$

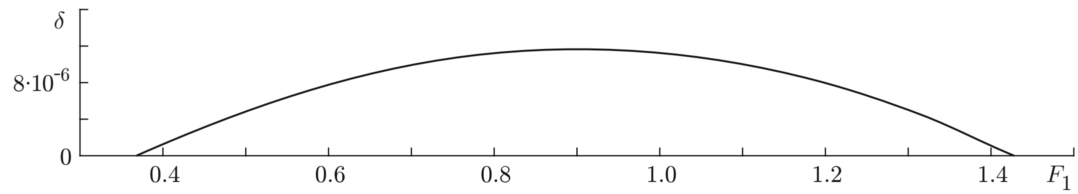

Fig. 3. Deviation of the bore diagram from the straight line AB (Fig. 2a).

Fig. 4. Solitary wave in the atmosphere.

we can conclude that the requirement of a positive value of the function P is a condition of the supercritical character of solitary waves of the principal mode with respect to the spectrum of linear waves.

Equation (8) also has solutions of the smooth bore type, which are obtained for the Froude numbers (F_1, F_2) generating a nonzero double root $s = a$ of the function $P(s; F_1, F_2)$. The dotted curve in Fig. 2 shows the segment AB of the bore diagram [set of points in the plane (F_1, F_2) to which the bore corresponds] with the points A ($F_1 = 0.4872$, $F_2 = 1.0572$) and B ($F_1 = 1.20702$, $F_2 = 0.7323$), which was calculated for the ratio of the layer thicknesses $r = 0.2$. The diagram is almost a straight line: Fig. 3 shows its deviation $\delta = \delta(F_1)$ from the segment of the straight line passing through the same points A and B. For the model with constant densities in the layer, which was developed in [2], the bore diagram is a straight line in the plane (F_1, F_2).

3. Example of a Solitary Wave in a Stratified Atmosphere. Results of modeling of internal solitary waves in the atmosphere, which were observed in summer 2004 above the lake Baikal surface (Fig. 4), by Eq. (8) are described below. The photograph displays a layer of foggy cold air immediately above the water surface (the water temperature on the lake surface does not exceed 7°C even in summer time). Expedition participants who frequently observed this atmospheric phenomenon estimated the thickness of the cooled air layer h_1 as 15–20 m. Finite-amplitude running solitary waves propagated along a clearly expressed interface between the cold heavy air (T = 10–12°C) and the lighter heated air (T = 18–20°C). Apparently, the sources of waves were individual perturbations arising due to descent of cold air masses from the tops of the near-shore ridges on the western shore of the lake.

To estimate the parameters of the warm air participating in the wave motion, we use available data on the properties of the lower atmosphere layers [5, 6]. The thickness of the atmosphere layer affected by the Earth's surface, which is the friction layer, is 1.0–1.5 km. In particular, the daily changes of the basic meteorological quantities are clearly expressed in this layer. In turn, a near-surface layer 50 to 100 m thick can be identified within the atmosphere layer considered; the vertical gradients of the basic physical quantities in this near-surface layer are greater than those in other layers by one or two orders of magnitude. Therefore, we use $h_2 \approx 70$ m as an estimate for the upper layer thickness in the two-layer medium model used. We neglect the motion of air in the atmosphere above this layer by applying the solid cover boundary condition at $y = h_2$. To determine the

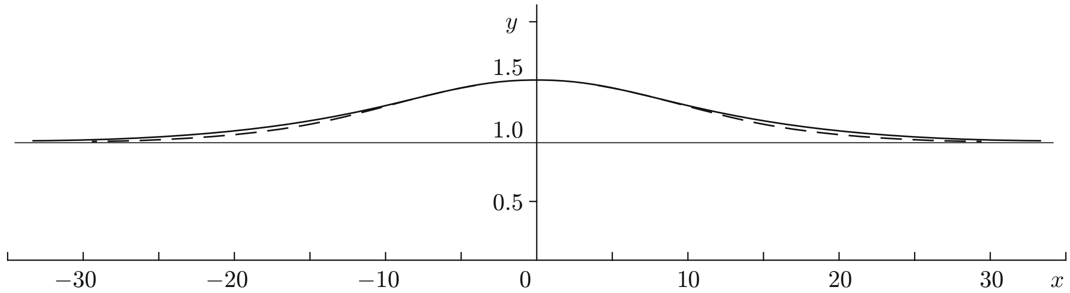

Fig. 5. Solitary wave profile obtained by solving Eq. (8) (dashed curve) and the contour of the wave shown in Fig. 4 (solid curve).

stratification parameters in the layer considered, we use the barometric formula [6]

$$\rho(y) = \rho_2 \, \mathrm{e}^{-gy/(R_c T_*)},$$

where $R_c = 287$ m^2/(sec$^2 \cdot$ K) is the specific gas constant for dry air (air humidity exerts only a weak effect on the exponent in the barometric formula), $T_* = 293$ K is the mean temperature in the layer considered, and ρ_2 is the air density at the lower boundary of this layer. Taking into account this information, we obtain the Boussinesq parameter $\sigma \approx 0.081$. Following [7], we use the density of air at a temperature of 12°C and humidity of 100%, which is equal to 1.181 kg/m^3, as ρ_1 and the density of air at a temperature of 20°C and humidity of 80%, which is equal to 1.170 kg/m^3, as ρ_2. Thus, we find the value of the second Boussinesq parameter $\mu = 0.094$, which is acceptable if model (8) is used. The solid curve in Fig. 5 shows the contour of the solitary wave presented in Fig. 4. The dashed curve in Fig. 5 is the calculated profile for the point E ($F_1 = 1.2685$, $F_2 = 0.5479$) in Fig. 2.

Conclusions. An equation of the second-order shallow water approximation for a weakly stratified two-layer fluid is derived. This equation describes solitary waves in a picnocline with a small difference in density. Calculations based on this model show that the approximation can be used to describe wave phenomena in the near-surface layer of a stratified atmosphere.

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